Completeness of Riemannian manifolds

(M^n, g) : Riemannian manifold (connected)

Define a distance on M as follow: for any $p, q \in M$.

$$d(p,q) := \inf \left\{ L(\gamma) : \begin{array}{l} \gamma : [0,1] \to M \text{ piecewise smooth} \\ \text{st } \gamma(0) = p,\ \gamma(1) = q \end{array} \right\}$$

FACT: (M, d) is a metric space.

Q: When is it "complete" as metric space?

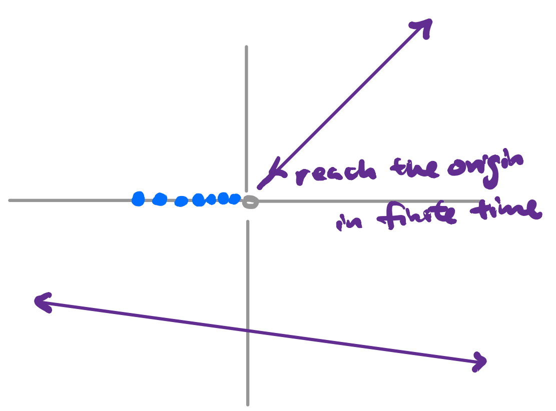
In fact, there is a more "differential-geometric" notion of "completeness".

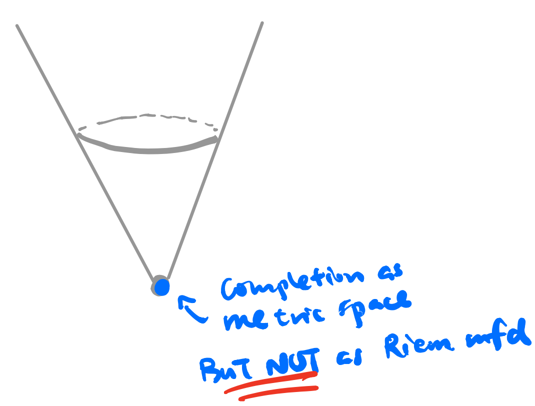
Def$^{\underline{n}}$: A Riem. mfd (M^n, g) is geodesically complete if any geodesic on M can be infinitely extended on both sides (ie defined on all of $\mathbb{R}$).

Examples:

(1) $\mathbb{R}^2 \setminus \{0\}$ w/. flat metric

(2) $M^2 =$ cone $\setminus$ {tip}. $\subseteq \mathbb{R}^3$

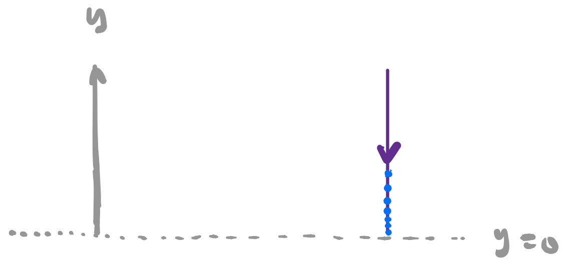




(1) NOT complete as metric space
NOT geodesically complete

(2) NOT complete as metric space
NOT geodesically complete

(3). $\mathbb{R}^2_+ := \{ (x,y) \in \mathbb{R}^2 \mid y > 0 \}$.



NOT (geodesically) complete w.r.t. g_{flat}

BUT (geodesically) complete w.r.t

hyperbolic metric $g_{hyp} := \frac{1}{y^2}(dx^2 + dy^2)$

Hopf-Rinow Theorem: Let (M^n, g) be a smooth Riem mfd.

THEN, the following are equivalent:

(1) (M^n, g) is geodesically complete

$\qquad\qquad\qquad\parallel$

(2) (M, d) is complete as a metric space

(3) The exponential map at p, $\exp_p$, is well-defined on the whole T_pM, for SOME $p \in M$.

(4) The exponential map at p, $\exp_p$, is well-defined on the whole T_pM, for ALL $p \in M$.

If any of the above holds, then

$$(*) \left[\begin{array}{l} \forall\, p, q \in M,\ \exists \text{ minimizing geodesic } \gamma : [0,1] \to M \\ \quad \text{st. } \gamma(0) = p,\ \gamma(1) = q \ \text{ and } \ L(\gamma) = d(p,q). \end{array} \right]$$

Proof: HW Exercise!